Two Limiting Values of the Capacitance of Symmetrical Rectangular Coaxial Strip Transmission Line

HENRY J. RIBLET, FELLOW, IEEE

Abstract—**This paper determines the first two terms in two different expansions for the total capacitance of rectangular coaxial strip transmission line which are of interest in an improved approximation for the characteristic impedance of rectangular coaxial line. For this purpose, expansions which express the total capacitance of the rectangular coaxial strip transmission line exactly and explicitly in terms of it's dimensions are introduced. As a by-product, it is shown how these expansions may be terminated after a few terms to obtain approximations of sufficient accuracy for most purposes.**

In the Appendix, certain results from the theory of elliptic functions, that are required in this paper but are not presented in the literature on this problem, are reviewed and in some cases extended.

I. INTRODUCTION

A PROCEDURE for determining the total capacitance of the rectangular coaxial structure in which the inner conductor is a strip of zero thickness located symmetrically inside a rectangular outer conductor, as shown in Figs. 1 and 2, was given many years ago by Magnus and Oberhettinger [1], [2]. Their procedure involves, among other things the solution of two transcendental equations, one of which is the inverse of the other. Although their treatment of the problem, based, as it is, on the parameter k of Jacobi's theory of elliptic functions, is adequate for numerical purposes, it is of no help in finding the limiting behavior of the capacitance of the rectangular coaxial strip transmission line which is the principal objective of this paper. For this purpose, it was found that the nome q of Jacobi's theory of theta functions is the useful parameter. In fact, using q as the fundamental parameter rather than k has substantial advantages even for numerical purposes.

In the first place, $q=\exp(-\pi K'/K)$ so that the parameter of the elliptic functions that appear, is given directly in terms of the shape of the outer conductor of the rectangular coaxial structure. Thus, the troublesome problem of solving the first transcendental equation of the earlier treatment is avoided altogether. Secondly, as will be shown, the total capacitance of the rectangular coaxial structure is given by the logarithm of a second nome q'_0 for which a number of terms of the convergence series in terms of k_0 are known. It is this latter fact which permits the expansion of the total capacitance of the rectangular coaxial structure directly in terms of it's dimensions, and leads to the derivation of the two limiting expressions which are the main objective of this paper. These limiting values are of interest because of the role that they play in improving the approximation for the characteristic impedance of rectangular coaxial line recently presented by Riblet [3]. As a useful by-product, general approximations for the capacitance of the rectangular coaxial strip transmission line, expressed directly in terms of it's dimensions, are given

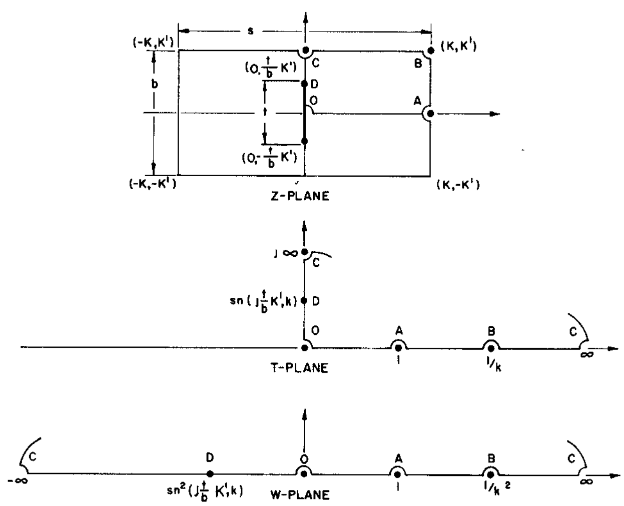

Fig. 1. Coaxial structure with vertical strip.

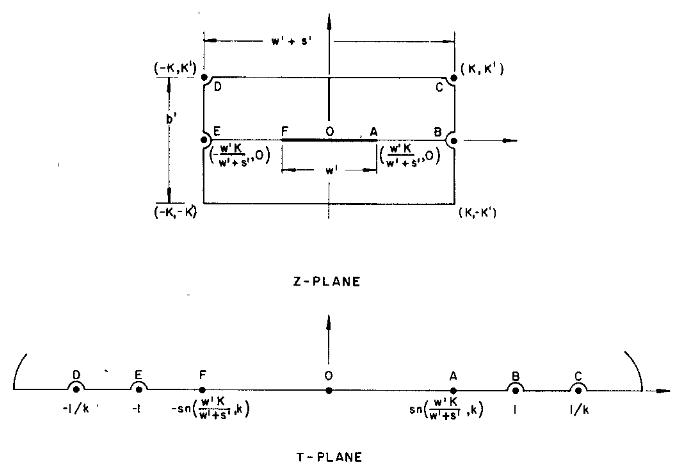

Fig. 2. Coaxial structure with horizontal strip.

Manuscript received September 18, 1980; revised January 13, 1981.

The author is with Microwave Development Laboratories, Inc., Natick, MA 01760.

0018-9480/81/0700-0661$00.75 ©1981 IEEE

which are of sufficient accuracy for most engineering purposes.

II. The Analysis

It is no restriction to assume that the outer rectangular conductor of the coaxial structure is viewed so that its horizontal dimensions is no less than its vertical dimensions as has been done in Figs. 1 and 2. Then, of course, the inner conductor may be vertical as shown in Fig. 1 or horizontal as in Fig. 2. Here the w, s, t, b notation used by Cohn [4] is employed where w is the width of the inner conductor and t is its thickness, while $w+s$ is the width of the outer conductor and b is its height. Then in Fig. 1, $w=0$ and the width of the outer conductor is s, while in Fig. 2 $t=0$ and the width of the outer conductor is $w+s$. In order to minimize the confusion, the dimensions in Fig. 2 have been given primes. If these quantities are related by the equations $s/b=1/(w'/b'+s'/b')$ and $t/b=w'/b'/(w'/b'+s'/b')$ then the figures are identical except for a 90° rotation.

The first limiting value to be obtained will be that of the total capacitance C_0 of the coaxial structure of Fig. 1 in the limit as $s/b\to\infty$. This case is considered first because it leads in a direct way to the solution given by Oberhettinger and Magnus.

Now it is basic to the theory of elliptic integrals that an elliptic integral of the first kind

$$\int_0^t \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

having the modulus k maps the upper half of the T-plane into the interior of the rectangle shown in the upper half of the Z-plane. It follows that the inverse of the elliptic integral of the first kind, Jacobi's elliptic function, $\mathrm{sn}(Z,k)$, maps the interior of the rectangle in the upper right-hand quadrant of the Z-plane into the upper right-hand quadrant of the T-plane. The transformation

$$T=\mathrm{sn}(Z,k) \tag{1}$$

then maps the point D, which falls at the end of the strip in the Z-plane onto the point D on the imaginary axis of the T-plane having the coordinates $(0,\mathrm{sn}(jtK'/b,k))$. Of course, it is assumed that K/K' is chosen equal to s/b. Now the further transformation

$$W=T^2 \tag{2}$$

maps the upper right hand quadrant of the T-plane into the upper half W-plane where corresponding points are denoted by the same letters. The point D of the W-plane now falls on the negative real axis and has the real coordinate, $\mathrm{sn}^2(jtK'/b,k)$. Clearly the total capacitance of the rectangular coaxial strip structure of the Z-plane is four times the capacitance of the segment DO, with respect to the infinite segment AC found in the upper half of the W-plane. As is well known [5], this capacitance is $K(k_0)/K'(k_0)$, where

$$k_0^2=\frac{(a-b)(c-d)}{(a-c)(b-d)} \tag{3}$$

with $a=\mathrm{sn}(jtK'/b,k)$, $b=0$, $c=1$, $d=\infty$. Since $\mathrm{sn}(jtK'/b,k)=j\,\mathrm{sn}(tK'/b,k')/\mathrm{cn}(tK'/b,k')$, one readily finds that $k_0^2=\mathrm{sn}^2(tK'/b,k')$. Thus

$$k_0=\mathrm{sn}(tK'/b,k') \tag{4}$$

and

$$k_0'=\mathrm{cn}(tK'/b,k').$$

Then the total capacitance C_0 of the rectangular coaxial strip structure is given, formally at least, by

$$C_0=4K(k_0)/K'(k_0) \tag{5}$$

where k_0 is defined by (4).

Equation (4) expresses k_0 in a form equivalent to that given by Oberhettinger and Magnus [1, p. 384] or [2, p. 64]. They recognize that "one has to solve the transcendental equation, $a/d=K'/K$, first"[1] and devote one page of their paper [1, p. 388] to a discussion of this problem.

Thus an unexpected result of the original investigation was the realization that this difficulty can be avoided by using the "nome" of Jacobi's theory of the theta functions, for the basic parameter. For complete consistency, perhaps, the k' of (4) should be replaced by q'. In any case, it is readily shown [6] that

$$\mathrm{sn}(tK'/b,k')=\frac{1+2q'+2q'^4+2q'^9+\cdots}{1+q'^2+q'^6+q'^{12}+\cdots}\cdot\frac{\sin\nu'+q'^2\sin3\nu'+q'^6\sin5\nu'+\cdots}{1-2q'\cos2\nu'+2q'^4\cos4\nu'-\cdots} \tag{6}$$

where $q'=\exp(-\pi K/K')=\exp(-\pi s/b)$ and $\nu'=\pi t/2b$. Also

$$\mathrm{cn}(tK'/b,k')=\frac{1-2q'+2q'^4-2q'^9+\cdots}{1+q'^2+q'^6+q'^{12}+\cdots}\cdot\frac{\cos\nu'+q'^2\cos3\nu'+q'^6\cos5\nu'+\cdots}{1-2q'\cos2\nu'+2q'^4\cos4\nu'-\cdots}. \tag{7}$$

Since $s/b>1$, $q'\leqslant 0.044$; and it is seen that (6) and (7) converge with extreme rapidity. Thus these equations express k_0 and k_0' directly in terms of the dimensions s, t, and b, of the rectangular coaxial stripline.

If we put $\alpha=\sin\nu'=\sin(\pi t/2b)$, and consider only fourth or lower powers of q'

$$k_0\doteq\frac{1+2q'+2q'^4}{1+2q'^2}\cdot\frac{\alpha(1-(3-4\alpha^2)q'^2)}{1-2(1-2\alpha^2)q'+2(8\alpha^4-8\alpha^2+1)q'^4}$$
$$\doteq\alpha\frac{1+2q'+(4\alpha^2-3)q'^2+(8\alpha^2-6)q'^3+2q'^4}{1+(4\alpha^2-2)q'+q'^2+(4\alpha^2-2)q'^3+(16\alpha^4-16\alpha^2+2)q'^4}. \tag{8}$$

For $s/b\geqslant1$, this approximation for k_0 is accurate to better than one part in 10^{-7} since it is readily shown that the

[1] English translation.

coefficients of the neglected terms in $\exp(-6\pi s/b)$ are less than 3 in absolute value. When (8) is expanded in a power series in σ

$$k_0=\alpha\big(1-4(\alpha^2-1)\exp(-\pi s/b) + (16\alpha^4-20\alpha^2+4)\exp(-2\pi s/b)+\cdots\big). \quad (9)$$

Similarly, if $\beta=\cos(\pi t/2b)$

$$k_0'=\beta\frac{1-2q'+(4\beta^2-3)q'^2-(8\beta^2-6)q'^3+2q'^4}{1-(4\beta^2-2)q'+q'^2-(4\beta^2-2)q'^3+(16\beta^4-16\beta^2+2)q'^4}. \quad (10)$$

Of course, the remarks made concerning the accuracy of (8) apply to (10). Also

$$k_0'=\beta\big(1+4(\beta^2-1)\exp(-\pi s/b) + (16\beta^4-20\beta^2+4)\exp(-2\pi s/b)+\cdots\big). \quad (11)$$

In a purely numerical sense, (6) and (7) are exact solutions of the problem and (8) and (10) are approximate solutions since $K(k_0)$ and $K'(k_0)$ may be obtained readily from tables [7] or from Landen's transformation [8]. Since we require, however, an expansion for the total capacitance C_0 of the structure in the form

$$C_0=A+B\exp(-\pi s/b)+\cdots \quad (12)$$

further analysis is required.

The relationship[2] between the modulus k and the nome q of the Jacobi theory can be expressed in the form [6, p. 486]

$$q=\epsilon'(1+2\epsilon'^4+15\epsilon'^8+\cdots) \quad (13)$$

where

$$\epsilon'=\frac{1}{2}\frac{1-\sqrt{k'}}{1+\sqrt{k'}}. \quad (14)$$

By taking the logarithm of q in (13), an expansion for K'/K directly in terms of ϵ' is obtained. Then

$$\frac{K'}{K}=-\frac{1}{\pi}\{\ln(\epsilon')+2\epsilon'^4+13\epsilon'^8+\cdots\}. \quad (15)$$

Now it is convenient to consider two cases. If $t/b\leqslant 0.5$, we learn from (11) that $k_{0'}\to\beta(1+4(\beta^2-1)\exp(-\pi s/b))$ as $s/b\to\infty$; and, in fact, $k_0'\geqslant 0.5-\delta$ where δ is arbitrarily small. Then surely $\epsilon'<0.044$ and the convergence of (15) is extremely rapid. In fact the error made by neglecting the second term in (15) is less than 0.001 percent.

Now

$$\sqrt{k_0'}\to\sqrt{\beta}\big(1+2(\beta^2-1)\exp(-\pi s/b)\big) \quad (16)$$

as $s/b\to\infty$, while

$$\epsilon'\to\frac{1}{2}\frac{1-\sqrt{\beta}}{1+\sqrt{\beta}}\big(1+4\sqrt{\beta}(1+\beta)\exp(-\pi s/b)\big) \quad (17)$$

in the same limit. Thus

$$\frac{K_0'}{K_0}\to-\frac{1}{\pi}\left\{\frac{1}{2}\ln\left(\frac{1-\sqrt{\beta}}{1+\sqrt{\beta}}\right)+4\sqrt{\beta}(1+\beta)\exp(-\pi s/b)\right\} \quad (18)$$

as $s/b\to\infty$. Finally

$$C_0\to -4\pi\left\{\frac{1}{\ln\left(\frac{1}{2}\frac{1-\sqrt{\beta}}{1+\sqrt{\beta}}\right)}-\frac{4\sqrt{\beta}(1+\beta)}{\ln^2\left(\frac{1}{2}\frac{1-\sqrt{\beta}}{1+\sqrt{\beta}}\right)}\exp(-\pi s/b)\right\}. \quad (19)$$

It should be observed that the coefficients in a series for C_0 in powers of $\exp(-\pi s/b)$ are themselves series resulting from the substitution of (17) in (15). The selection of β, however, ensures that all terms containing ϵ'^4 are negligible for the present purpose.

If, on the other hand, $t/b\geqslant 0.5$, from (9) $k_0\to(1-4(\alpha^2-1)\exp(-\pi s/b))$ as $s/b\to\infty$; and in fact $k_0\geqslant 0.5$. Thus

$$\sqrt{k_0}\to\sqrt{\alpha}\big(1-2(\alpha^2-1)\exp(-\pi s/b)\big) \quad (20)$$

as $s/b\to\infty$, while

$$\epsilon\to\frac{1}{2}\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\big(1-4\sqrt{\alpha}(1+\alpha)\exp(-\pi s/b)\big). \quad (21)$$

Moreover

$$\frac{K_0}{K_0'}\to-\frac{1}{\pi}\left\{\ln\left(\frac{1}{2}\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)-4\sqrt{\alpha}(1+\alpha)\exp(-\pi s/b)\right\} \quad (22)$$

in this limit. Finally

$$C_0\to-\frac{4}{\pi}\left\{\ln\left(\frac{1}{2}\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)-4\sqrt{\alpha}(1+\alpha)\exp(-\pi s/b)\right\} \quad (23)$$

as $s/b\to\infty$.

The accuracy of (19) and (23) can be estimated by comparing the values of their two coefficients at their point of common validity; namely when $\alpha=\beta=0.5$. It will be found that, even at this point where they can be expected to be least accurate, they agree to within 0.001 percent.

The treatment of the case shown in Fig. 2, where it is assumed that $(w'+s')/b'>1$, proceeds differently. As has been seen

$$T=\mathrm{sn}(Z,k) \quad (24)$$

[2] This relationship is discussed in greater detail in the Appendix.

maps the interior of the rectangle in the upper half of the Z-plane into the upper half of the T-plane. Here corresponding points in the two planes are given the same letter designation. Clearly (24) maps the portion of the real axis of the Z-plane onto that segment of the real axis of the T-plane between $\pm\mathrm{sn}(w'K'/(w'+s'), k)$. Here the modulus k is determined by $K(k)/K'(k)=(w'+s')/b'$.

To determine the total capacitance C_0, of the rectangular coaxial structure, we have only to double the capacitance in the upper half of the T-plane between the line segments, FA and BE. Thus

$$C=2K'(k_0)/K(k_0) \tag{25}$$

where

$$k_0^2=(a-b)(c-d)/(a-c)(b-d). \tag{26}$$

In this case, however, $a=-1$, $b=-\mathrm{sn}(w'K/(w'+s'), k)$, $c=\mathrm{sn}(w'K/(w'+s'), k)$ and $d=1$. Then substituting, it is readily found that

$$k_0=\frac{1-\mathrm{sn}(w'K'/b', k)}{1+\mathrm{sn}(w'K'/b', k)} \tag{27}$$

if it is recalled that $K/(w'+s')=K'/b'$.

Careful consideration will show that the $\mathrm{sn}(w'K'/b', k)$ of (27) is exactly the same as the k_0 of (4) if the geometries of the two cases are identical except for a 90° rotation. Thus the two values of k_0 are related by a modular transformation [9]. This explains the difference between the two formulas for C_0, (5) and (25). In the present application, (27) is preferable to (4) even though it appears to be more complicated.

As before, k_0 can be determined from (27) by first finding k from the given value of $(w'+s')b'$. This troublesome step can be avoided, however, by expanding $\mathrm{sn}(w'K'/b', k)$ directly in terms of the dimensions of the given rectangular coaxial structure. It is known [10] that

$$\mathrm{sn}(w'K'/b', k)=-\frac{1+2q'+2q'^4+2q'^9+\cdots}{1-2q'+2q'^4 2q'^9+\cdots}\cdot\frac{\sinh\nu'-q'^2\sinh 3\nu'+q'^6\sinh 5\nu'+\cdots}{\cosh\nu'+q'^2\cosh 3\nu'+q'^6\cosh 5\nu'+\cdots} \tag{28}$$

where $q'=\exp(-\pi K/K')=\exp(-\pi(w'+s')/b')$, and $\nu'=\pi w'K'/2K'b'=w'/2b'$. Since $(w'+s')/b'\geqslant 1$, $q'<0.044$ and the convergence of the terms in (28) is extremely rapid, since the exponents of the q' increase much more rapidly than the coefficients of ν'. If $\omega=\exp(-\pi w/b)$, and $\sigma=\exp(-\pi s/b)$, the hyperbolic functions are written in exponential form and the values of q' and ν' are substituted in (28)

$$\mathrm{sn}(w'K'/b', k) = \frac{1-(1-\sigma)^2\omega-2\sigma(1+\sigma^2)\omega^2+\sigma^2(1+\sigma^2)^2\omega^4+2\sigma^3(1-\sigma-\sigma^3+\sigma^4)\omega^5+2\sigma^6(1+\sigma^4)\omega^8-\sigma^6(1-\sigma^3)^2\omega^9+\cdots}{1+(1-\sigma)^2\omega-2\sigma(1+\sigma^2)\omega^2+\sigma^2(1+\sigma^2)^2\omega^4-2\sigma^3(1-\sigma-\sigma^3+\sigma^4)\omega^5+2\sigma^6(1+\sigma^4)\omega^8+\sigma^6(1-\sigma^3)^2\omega^9+\cdots} \tag{29}$$

where the neglected terms are of tenth or higher power in $\exp(-\pi w/b)$ and of seventh or higher power in $\exp(-\pi s/b)$. Substituting in (30)

$$k_0=(1-\sigma)^2\omega\cdot\frac{1-2\sigma^3(1+\sigma+\sigma^2)\omega^4+\sigma^6(1+\sigma+\sigma^2)^2\omega^8+\cdots}{1-2\sigma(1+\sigma^2)\omega^2+\sigma^2(1+\sigma^2)^2\omega^4+2\sigma^6(1+\sigma^4)\omega^8+\cdots} \tag{30}$$

where the neglected terms are of tenth or higher power in $\exp(-\pi w/b)$ and of seventh or higher power in $\exp(-\pi s/b)$. The terms in $\exp(-8\pi w/b)$ contain the factor, $\exp(-6\pi(w'+s')/b')$, which is less than 10^{-8} because of the restriction placed on $(w'+s')/b'$. Thus these terms can be neglected with an error less than 0.001 percent. Then

$$k_0 \doteq (1-\sigma)^2\omega\frac{1-2\sigma^3(1+\sigma+\sigma^2)\omega^4}{\left(1-\sigma(1+\sigma^2)\omega^2\right)^2}. \tag{31}$$

Of course, having found k_0, the determination of C_0 numerically follows from equations like (13), (14), and (15). If $k_0>\sqrt{0.5}$, one immediately determines q'; and, if $k<\sqrt{0.5}$ one first finds k_0' and then q.

A principal objective of this paper is the determination of the first two terms of an expansion of C_0 in powers of $\exp(-\pi w/b)$. With this in mind, it is found from (31) that

$$k_0=(1-\exp(-\pi s/b))^2\exp(-\pi w/b)\{1+2(\exp(-\pi s/b)+\exp(-2\pi s/b))\cdot\exp(-2\pi w/b)+\cdots\}. \tag{32}$$

Now as $w/b\to\infty$, $k_0\to 0$, with the consequence that (13) converges too slowly to be useful. Instead (32) is substituted directly in an expansion for C_0 derived in the Appendix

$$C_0=\frac{2}{\pi}\left\{\ln\left(\frac{16}{k_0^2}\right)-\left(\frac{1}{2}k_0^2+\frac{13}{64}k_0^2+\cdots\right)\right\}. \tag{33}$$

Finally

$$C_0=4\frac{w}{b}+\frac{8}{\pi}\{\ln(2)-\ln(1-\exp(-\pi s/b))\}-\frac{1}{\pi}(1+\exp(-\pi s/b))^4\exp(-2\pi w/b)+\cdots. \tag{34}$$

III. Conclusions

In the course of determining certain of the limiting values of the capacitances of rectangular coaxial strip transmission line, it was discovered that the capacitance of this form of transmission line can be found from rapidly convergent series whose terms are elementary functions expressible directly in terms of the dimensions of the structure. In fact, the convergence is so rapid that only a few terms of the series are required to obtain accuracies of the order of 0.001 percent or better in all cases.

APPENDIX

The determination of the capacitance of one segment of the real axis with respect to another is usually accomplished by a suitable conformal mapping of the upper half plane into the interior of a rectangle whose capacitance is given by the ratio of it's dimensions since it can be embedded in an infinite parallel plate capacitor. The cross ratio of the four end points of the two segments on the real axis determine the modulus k of the elliptic integral of the first kind which maps them on the opposite sides of a rectangle, the ratio of whose dimensions is $K(k)/K'(k)$. Thus the problem of finding the required capacitance is the problem of determining $K(k)/K'(k)$ given k. It is the object of this Appendix to discuss this problem and to give some possibly new results. By definition, $q' = \exp(-\pi K/K')$ so that the problem is also that of finding q' given k. Of course, the relationship between q' and k, the nome and the modulus, is fundamental to the theory of elliptic functions. It is given by Jacobi [11]

$$\sqrt{k'} = \frac{1-2q+2q^4-2q^9+\cdots}{1+2q+2q^4+2q^9+\cdots} \qquad \text{(I)}$$

with a similar relationship between k and q'. Moreover since $k^2+k'^2=1$ and $\ln(q) . \ln(q')=\pi^2$, either q or $q' < \exp(-\pi)$; and for a given value of q, one can always find the modulus using very rapidly convergent series.

The important fact for our problem is that (I) can be inverted and q expressed in terms of k'. In fact, if

$$\epsilon' = \frac{1}{2}\,\frac{1-\sqrt{k'}}{1+\sqrt{k'}} \qquad \text{(II)}$$

then

$$q = \epsilon' + 2\epsilon'^5 + 15\epsilon'^9 + 150\epsilon'^{13} + 1707\epsilon'^{17} + 20910\epsilon'^{21} + \cdots. \qquad \text{(III)}$$

Whittaker and Watson [6, p. 486] have discussed the derivation of this equation, shown that it is convergent for $\epsilon' < 0.5$ and presented the first four terms. The fifth coefficient in (III) may be found in [2, p. 19]. The writer does not know of any general formula for these coefficients but he has programmed their derivation on a digital computer and obtained additional terms. They increase in size by a factor slightly less than 16 so the convergence of (III) is very slow for small values of k'. By taking the logarithm of (III), the following expansion for K'/K is obtained[3]

[3] Tippet and Chang [12] have used the first term in this expansion in their approximation for the capacitance of rectangular coaxial strip transmission line.

$$\frac{K'}{K} = -\frac{1}{\pi}\left\{\ln(\epsilon') + 2\epsilon'^4 + 13\epsilon'^8 + \frac{368}{3}\epsilon'^{12} + \frac{2701}{2}\epsilon'^{16} + \frac{80912}{5}\epsilon'^{20} + \cdots\right\}. \qquad \text{(IV)}$$

For some purposes, this expansion has the advantage that it obviates the need to calculate q before finding the capacitance. In addition, it's convergence appears to be slightly more rapid than that of (III).

A problem arises if k is given by an expansion whose values of interest are so small that the convergence of (III) is too slow. Instead of finding an equivalent expansion for k', one may replace k' in (II) by $\sqrt{1-k^2}$. Then

$$\epsilon' = \frac{1}{2}\,\frac{1-(1-k^2)^{1/4}}{1+(1-k^2)^{1/4}}. \qquad \text{(V)}$$

This ϵ' can be expanded in a power series in k which can be substituted in (III) to yield a power series for q in terms of k. When the logarithm of this series is found an expansion for K'/K directly in terms of k is the result. It has been determined in this way that

$$\frac{K'}{K} = \frac{1}{\pi}\left\{\ln\left(\frac{16}{k^2}\right) - \left(\frac{1}{2}k^2 + \frac{13}{64}k^4 + \frac{23}{192}k^6 + \frac{2701}{3276}k^8 + \cdots\right)\right\}. \qquad \text{(VI)}$$

The first two terms of this equation is the familiar approximation for K'/K given in many texts on elliptic functions. For example, see Bowman [5, p. 22].

In order to estimate the accuracy of the expansions in (IV) and (VI) let $k=0.18$. If this substituted in (IV), $K'/K=0.50787429515$. On the other hand, if this value of k is substituted directly in (VI), $K'/K=0.50787429496$. These values differ only by 1.9×10^{-10}. Of course (IV) should be used for larger values of k and (VI) for smaller values. Clearly (IV) and (VI) will determine the relative capacitance of two segments on the real axis with an accuracy sufficient for most engineering purposes. Of course, as long as k and k' are readily available, (II) and (IV) will always find K'/K with great ease and accuracy since either ϵ or $\epsilon' < 0.044$.

The use of (IV) to determine K/K', even when $k > \sqrt{0.5}$, can be troublesome if k is so close to unity that the digital computer used cannot accurately distinguish it from unity or if it is given by a series which approaches unity. In these cases it is convenient to have an expansion for K/K' directly in terms of $1-k$. As in (V) when ϵ' is expanded as a series in $1-k$ and the result substituted in an equation like (IV)

$$\frac{K}{K'} = -\frac{1}{\pi}\left\{\log\left(\frac{1-k}{8}\right) + \frac{1}{2}(1-k) + \frac{3}{16}(1-k)^2 + \frac{5}{48}(1-k)^3 + \cdots\right\}. \qquad \text{(VII)}$$

REFERENCES

[1] W. Magnus and F. Oberhettinger, "Die berechnung des wellenwiderstandes," Arch. Elektrotech., 37 Bund, Heft 8, pp. 386–389, 1943.

[2] F. Oberhettinger and W. Mangus, *Anwendung Der Elliptischen Funktionen in Physik and Technik*. Berlin, Germany, Springer, 1949, pp. 62–65.

[3] H. J. Riblet, "Upper limits on the error of an improved approximation for the characteristic impedance of rectangular coaxial line," *IEEE Trans. Microwave Theory Tech*, (Short Papers), vol. MTT-28, pp. 666–667, June 1980.

[4] S. B. Cohn, "Characteristic impedance of the shielded-strip transmission line," *IRE Trans. Microwave Theory Tech*., Vol. MTT-2, pp. 52–57; July, 1954.

[5] F. Bowman, *Introduction to Elliptic Functions with Applications*. New York: Dover, 1961, pp. 56–59.

[6] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*. Cambridge, MA: University Press, 1940, pp. 492, 479, 464.

[7] L. M. Milne-Thompson, *Jacobian Elliptic Function Tables*. New York: Dover, 1950, pp. 106–109.

[8] M. Abromowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions* (Applied Mathematics Series 55). Washington, DC: NBS, 1964, pp. 571, 578, 598.

[9] A. Cayley, *An Elementary Treatise on Elliptic Functions*. New York: Dover, 1961, p. 180.

[10] E. B. Wilson, *Advanced Calculus*. Boston: Ginn and Company, 1911, pp. 471, 506.

[11] C. W. Borchardt, *Jacobi's Gesammelte Werke*. Berlin: G. Reimer, 1881, Bd I, p. 236, Eq. (11).

[12] J. C. Tippet and D. C. Chang, "A new approximation for the capacitance of a rectangular coaxial strip transmission line," *IEEE Trans. Microwave Theory Tech*, (Short Papers), vol. MTT-24, pp. 602–604, Sept. 1975.

On the Radiation from Microstrip Discontinuities

MOHAMAD D. ABOUZAHRA, STUDENT MEMBER, IEEE

Abstract—**The method of an earlier paper by Lewin is used to calculate, more accurately, the radiated power from a microstrip termination. The substrate dielectric constant ϵ is used instead of the effective dielectric constant ϵ_e in the polarization term. The open-circuit, short-circuit, and matched coaxial terminations are deduced as particular cases of the general termination. On comparison with Lewin's results, differences of up to 30 percent have been found, but the differences are much smaller for the larger values of the actual relative dielectric constant ϵ. Curves show that the short-circuit termination radiates less than a quarter that of the open circuit, and can be considered as a means of reducing losses in microstrip resonators. The parallel post configuration is also considered.**

I. INTRODUCTION

IN AN earlier paper [1], Lewin used the far-field Poynting vector method to calculate the radiation from microstrip discontinuities. In the course of his calculation, and in order to account for the leakage of the field into the air above the strip, Lewin used the effective relative dielectric constant ϵ_e in both the propagation constant and the polarization part of the calculation. Using a completely different type of analysis, utilizing Fourier transforms and a more involved treatment of the microstrip configuration, Van der Pauw [2] derived a more accurate expression for the open-circuit case. Recently, the calculations of Lewin for the open-circuit and matched termination were repeated [3] with ϵ_e replaced by ϵ in the polarization term. The results for the open-circuit case agree with that of Van der Pauw. From [3] it was discovered that the main difference between the results of [1] and [2] was not from the radically different treatment, but from the use of ϵ_e rather than ϵ in the calculation of the contribution of the dielectric polarization to the radiated fields. In this paper, Lewin's method will be extended to derive a more accurate expression for the general termination case from which results for three particular cases will be deduced. These are the open-circuit, short-circuit, and the matched coaxial termination. The parallel post configuration is also reconsidered and a more accurate expression for the radiated fields and the radiated power are derived.

Manuscript received September 30, 1980; revised February 16, 1981.

The author is with the Electromagnetics Laboratory, Department of Electrical Engineering, University of Colorado, Boulder, CO 80309.

II. ANALYSIS

Fig. 1 shows the microstrip configuration as well as the coordinate system used in this paper. In this analysis, a new scheme of notation, different from that of [3], will be adopted. By replacing ϵ_e (ϵ in [3]), the *effective* dielectric constant, by ϵ (ϵ^* in [3]), the *actual* dielectric constant, in the polarization term and then calculating the far-field Hertzian vector, the far-field expressions for the mis-

0018-9480/81/0700-0666$00.75 ©1981 IEEE